

Cointegrated Limited Time Series

Cointegrazione in Serie Storiche Limitate

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Riassunto: In questo lavoro sono discussi i concetti di integrazione e cointegrazione in serie storiche “limitate”, il cui campo di variazione è soggetto a vincoli. Si mostra, sulla base di una nuova teoria asintotica (basata sul concetto di moto Browniano regolato) e di alcune simulazioni Monte Carlo, come la presenza di tali vincoli invalidi l’uso dei tradizionali test di integrazione e di cointegrazione.

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1. Introduction

It is somewhat surprising that many of the most well-known applications of unit root and cointegration techniques involve variables which cannot be integrated of order 1 $[I(1)]$ as they are “limited”, in the sense that their range is constrained by fixed bounds. Some examples are: (i) money demand equations, involving (among other variables) the nominal interest rate, which is forced to be non-negative and cannot be $I(1)$ in the usual sense; (ii) Fisher relations, involving inflation and nominal interest rates; (iii) the expectation hypothesis of the term structure, involving yields to maturity, obtained as the difference between two nominal interest rates; (iv) long-run Almost Ideal Demand Systems, involving expenditure shares, bounded between 0 and 1; (v) unemployment, with the unemployment rate being usually modeled as $I(1)$, in contrast with the fact that the unemployment rate is bounded both above and below; (vi) managed exchange rates, with nominal rates kept within a target zones.

The notion of (co-)integration becomes ambiguous when applied to limited time series. Specifically, it is not obvious either how to interpret cointegrating relations involving limited time series or whether the notions of integration and cointegration can in fact be extended to cases involving limited time series. Indeed, a time series which is bounded cannot strictly be an integrated process, since the latter is theoretically unbounded, with the tendency of diverging infinity as the time span diverges. Still, there are many economic and financial time series that, despite the fact of being bounded, are well represented as integrated processes, cf examples (i)-(vi) above and Nicolau (2002). Similarly, it is not obvious how the well-known Engle-Granger (1987) notion of cointegration can be extended to cover multiple time series with some components being bounded. Yet, all examples (i)-(vi) above report cases of bounded variables sharing a common long run comovement with some other variables, either bounded or unbounded.

In this framework, our aim is to extend the existing literature by focusing on the theory of (co-)integration for possibly bounded time series. Specifically, by drawing on some recent results on bounded scalar $I(1)$ processes established in Cavaliere (2005), we wish to fill the gap in the literature through the following steps. First, we aim at

showing how the definition of (co-)integration can be generalized in order to apply to time series which might be bounded. Second, we aim at explaining how the presence of bounds affects the properties of (co-)integration techniques. Third, we aim at explaining why in some cases the bounds can reasonably be neglected and therefore standard co-integration methods continue to be appropriate.

2. Preliminaries

Before discussing cointegration in limited time series, we briefly recall the notion of limited I(1) process. As in Cavaliere (2005), a time series X_t is said to be *limited*, or *bounded* I(1) [BI(1)], with bounds at $\underline{b} < \bar{b}$, if it satisfies $\Delta X_t = \Delta S_t + \underline{\xi}_t - \bar{\xi}_t$, where S_t is a standard I(1) process with long run variance λ^2 and $\underline{\xi}_t, \bar{\xi}_t$ are two regulator processes (see Harrison, 1985) which constraint X_t to lie within a given interval $[\underline{b}, \bar{b}]$. Under some technical conditions (see Cavaliere, 2005, Assumption A), and if $\underline{b} = \underline{c}\lambda T^{1/2}, \bar{b} = \bar{c}\lambda T^{1/2}$, then the following invariance principle holds:

$$[1] \quad T^{-1/2}X_{[T]} \Rightarrow \lambda B_{\underline{c}}^{\bar{c}}(\bullet) \quad (\text{weakly on } [0,1], \text{ as } T \rightarrow \infty)$$

$B_{\underline{c}}^{\bar{c}}(\bullet)$ being a *regulated Brownian motion* [BM]. Obviously, for $-\underline{b} = \bar{b} = \infty, X_t = S_t$ is I(1) and $B_{\underline{c}}^{\bar{c}}(\bullet) = B(\bullet)$, a standard BM. It is worth noting that since in the BI(1) case $X_t = S_t + L_t - U_t$, where $L_t := \sum_{i=1}^t \underline{\xi}_i, U_t := \sum_{i=1}^t \bar{\xi}_i$ are non-stationary (see Cavaliere, 2005), there are two sources of possible non-stationarity in the data: the I(1) trend S_t and the cumulated regulators L_t, U_t . A vector bounded I(1) process can be defined similarly.

Definition 1. Let $\mathbf{X}_t = (X_{1t}, \dots, X_{pt})'$ and let $\mathbf{S}_t$ be a p -vector I(1) process with full-rank long run covariance matrix $\mathbf{\Lambda}$. If for each $i=1, \dots, p$, $\Delta X_{it} = \Delta S_{it} + \underline{\xi}_{it} - \bar{\xi}_{it}$ where the regulators $\underline{\xi}_{it}, \bar{\xi}_{it}$ keep X_{it} within the interval $[\underline{b}_i, \bar{b}_i]$, $\mathbf{X}_t$ is said to be vector BI(1).

The definition above does not require all components of $\mathbf{X}_t$ to be bounded. Indeed, as any of the $\underline{b}_i, \bar{b}_i$ can be infinity, it allows for both (one-sided and two-sided) bounded and unbounded components. The following invariance principle [IP] holds:

$$[2] \quad T^{-1/2}\mathbf{X}_{[T]} \Rightarrow \mathbf{\Lambda}^{1/2}\bar{\mathbf{B}}(\cdot) \quad (\text{weakly on } [0,1], \text{ as } T \rightarrow \infty)$$

where $\bar{\mathbf{B}}(\cdot)$ is a vector of independent regulated BMs. If all $\underline{b}_i, \bar{b}_i$ are infinity, $\bar{\mathbf{B}}(\cdot) = \mathbf{B}(\cdot)$ and the standard multivariate IP $T^{-1/2}\mathbf{X}_{[T]} \Rightarrow \mathbf{\Lambda}^{1/2}\mathbf{B}(\cdot)$ holds; cf Davidson (1994).

3. Cointegration among limited time series

Consider a (possibly bounded) k -dimensional process $\mathbf{X}_t$ generated by the autoregression

$$[3] \quad \Delta \mathbf{X}_t = \mathbf{\Pi} \mathbf{X}_{t-1} + \mathbf{u}_t + \underline{\xi}_t - \bar{\xi}_t,$$

where $\underline{\xi}_t = (\underline{\xi}_{it})$, $\bar{\xi}_t = (\bar{\xi}_{it})$ are the two-sided regulators; cf Section 2. For simplicity we assume that there are no deterministic components, that $\mathbf{u}_t$ is $\text{WN}(\mathbf{0}, \mathbf{\Lambda})$ and that $\mathbf{\Pi}$ satisfies the so-called I(1) conditions, cf Johansen (1996). As in the standard case, we say that the bounded process $\{\mathbf{X}_t\}$ is *cointegrated* if $\mathbf{\Pi}$ has reduced rank r , $0 < r < k$; the no-cointegration case corresponds to the null hypothesis $r=0$, i.e. $\mathbf{\Pi}=\mathbf{0}$.

We now discuss the properties of two well-known tests of no cointegration under the null hypothesis, when some of the components of $\mathbf{X}_t$ are bounded. First, we consider a standard residual-based test for the null of no cointegration, cf Phillips and Ouliaris (1990) [PO], which in its simplest form is based on the statistic $Z_{\hat{\alpha}} := T\hat{\alpha}$, $(\hat{\alpha}+1)$ being the sample first-order autoregressive coefficient of $\hat{u}_t$, the residuals of the OLS projection of Y_t onto the space spanned by $\mathbf{Z}_t$, $\mathbf{X}_t = (Y_t, \mathbf{Z}_t')$. The relevant critical values in the absence of bounds are tabulated in PO. Second, we consider the so-called trace test, based on the statistic $LR = -T \sum_{i=1}^k \log(1 - \hat{\lambda}_i)$; see Johansen (1996). In the case of bounds, the asymptotic distributions of $Z_{\hat{\alpha}}$ and LR are given in the next theorem.

Theorem 1. Under the d.g.p. [3], if the null hypothesis holds then, as $T \rightarrow \infty$,

$$[4] \quad Z_{\hat{\alpha}} \Rightarrow \frac{\int_0^1 Q^*(s) dQ^*(s)}{\int_0^1 Q^*(s)^2 ds}$$

$$[5] \quad LR \Rightarrow \text{tr} \left(\int_0^1 d\bar{B}(s) \bar{B}(s)' \left(\int_0^1 \bar{B}(s) \bar{B}(s)' ds \right)^{-1} \int_0^1 \bar{B}(s) d\bar{B}(s)' \right)$$

where $Q_{01}^*(s) := \bar{B}_0(s) - \left(\int_0^1 \bar{B}_0(r) \bar{B}_1(r)' dr \right) \left(\int_0^1 \bar{B}_1(r) \bar{B}_1(r)' dr \right)^{-1} \bar{B}_1(s)$ and $\bar{B} = (\bar{B}_0, \bar{B}_1)'$ is a vector of k independent regulated BMs, $\bar{B}_0$ being a scalar process.

Theorem 1 shows that, when applied to bounded time series, standard cointegration tests are no longer valid. Specifically, the asymptotic null distributions, instead of depending on standard BMs, now depend on *regulated* BMs.

Two facts are worth noting. First, if the time series involved are unbounded, then the limiting processes $\bar{B}_0, \bar{B}$ become two standard BMs, and the usual distribution theory (PO, Th.4.1; Johansen, 1996, Th.11.1, *inter alia*) follows. Second, if the bounds of the components of X_t are sufficiently far away, then the probability of breaching the bounds is roughly zero and $\bar{B}$ is well approximated by a standard vector BM. In this case, standard asymptotic theory continues to provide a useful approximation.

In order to evaluate the impact of the bounds, we perform a simple Monte Carlo experiment based on a bivariate data generating process where, without loss of generality, we set $\mathbf{\Lambda}=\mathbf{I}_2$. Both components start at 0 and are bounded at $\underline{b}_i, \bar{b}_i$, $\underline{b}_i = c_i \sqrt{T}$, $\bar{b}_i = \bar{c}_i \sqrt{T}$, $i=1,2$. For the PO test we present the test based on regressing X_1

Table 1: Size of some tests of the no cointegration hypothesis in the presence of bounds

		two bounded components					one bounded component				
c_1	∞	1.50	0.70	0.50	0.40	0.30	1.50	0.70	0.50	0.40	0.30
c_2	∞	1.50	0.70	0.50	0.40	0.30	∞	∞	∞	∞	∞
LR	0.05	0.04	0.05	0.10	0.25	0.94	0.04	0.05	0.06	0.07	0.27
$Z_{\hat{\alpha}}(1 2)$	0.05	0.05	0.07	0.10	0.16	0.57	0.05	0.07	0.10	0.17	0.58
$Z_{\hat{\alpha}}(2 1)$	0.05	0.05	0.07	0.11	0.17	0.59	0.05	0.05	0.05	0.05	0.04

Results are based on 10,000 MC iterations with the limiting processes approximated over a grid of $T=2,500$ points. Nominal level: 0.05.

on X_2 (denoted as $Z_{\hat{\alpha}}(1|2)$) as well as the test based on regressing X_2 on X_1 ($Z_{\hat{\alpha}}(2|1)$). The results are reported in Table 1. In the two-bound case, as the bounds become closer to the starting value of the process, the size of the cointegration tests becomes higher than the nominal level, hence leading to spurious rejection of the hypothesis of no cointegration. Note that for $c_1=c_2=0.3$, size is over 50% for the PO test, and over 80% for the trace test. In the one-bound case, size distortions are much less pronounced for the trace test. Interestingly, the PO test has size up to 50% when the bounded variable is placed on the left hand side of the auxiliary regression, while its size is correct when the bounded variable is placed on the right hand side.

4. Conclusions

In this note we have shown that the application of standard co-integration tests to *bounded* time series (e.g., nominal interest rates, unemployment rates, expenditure shares, and so forth) is no longer valid. Specifically, the presence of bounds might inflate the size of the tests. This result extends those given in Cavaliere (2005), where it is shown under which conditions unit root tests can be applied to bounded time series. The results given here constitute the basis of new cointegration tests which are robust to the presence of bounds.

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